ALGEBRAIC THEORIES

THOMAS READ

Supervised by
MARTIN HYLAND

Part 1

What are algebraic theories?

What is an algebraic theory?

- A group is a set G equipped with

$m : G \times G \longrightarrow G$

satisfying

$\forall x, y, z \in G, \quad m(m(x,y),z) = m(x, m(y,z))$

$\exists e \in G$ s.t. i) $\forall x \in G, \; m(e,x) = x = m(x,e)$

ii) $\forall x \in G, \exists x' \in G$ s.t.

$m(x', x) = e = m(x, x')$

What is an algebraic theory?

- A group is a set G equipped with

$m : G \times G \longrightarrow G$

$i : G \longrightarrow G$

$e \in G$

satisfying

$\forall x, y, z \in G \quad m(m(x,y),z) = m(x, m(y,z))$

$\forall x \in G \quad m(i(x),x) = e$

$\forall x \in G \quad m(x,i(x)) = e$

$\forall x \in G \quad m(e,x) = x$

$\forall x \in G \quad m(x,e) = x$

What is an algebraic theory?

- A **group** is a set G equipped with

$$m : G^2 \longrightarrow G$$
$$i : G^1 \longrightarrow G$$
$$e : G^0 \longrightarrow G$$

satisfying

$$\forall x, y, z \in G \quad m(m(x,y),z) = m(x, m(y,z))$$
$$\forall x \in G \quad m(i(x),x) = e$$
$$\forall x \in G \quad m(x,i(x)) = e$$
$$\forall x \in G \quad m(e,x) = x$$
$$\forall x \in G \quad m(x,e) = x$$

What is an algebraic theory?

- An equational theory consists of

i) A set Ω of function symbols

ii) An arity function $\alpha : \Omega \longrightarrow \mathbb{N}_0$

iii) A set T of equations built from the function symbols and variables $x, y, z, \ldots$

e.g. $\Omega = \{m, i, e\}$ $\quad \alpha(m) = 2 \quad \alpha(i) = 1$
$\alpha(e) = 0$

$$T = \{m(m(x, y), z) = m(x, m(y, z)),$$
$$m(i(x), x) = e,$$
$$\ldots\}$$

What is an algebraic theory?

- A model of an equational theory is

i) A set A

ii) For each function symbol $f \in \Omega$, an interpretation

$$|f| : A^{\alpha(f)} \longrightarrow A$$

iii) such that "the equations hold"

e.g. for groups, have

$\forall x, y, z \in A,\ |m|(|m|(x, y), z) = |m|(x, |m|(y, z))$

and so on...

What is an algebraic theory?

- Examples: Groups, Rings, Lattices

- Non-examples: Fields

  Why? Equations must be universally quantified

  (but takes a little more work to prove no axiomatisation works)

Universal algebra

- Want to study equational theories as objects in their own right
- Problem: "same" theory can be presented by different operations and equations
- Solution: Abstract clones or Lawvere theories

Idea: Don't treat some operations as "basic" or "special"

Instead consider the set of <u>all</u> operations, and describe how they compose

Part 2

What did I do?

Tensors of thys

- The tensor product is a way to combine two Lawvere theories and get a new one
- A model of $L_1 \otimes L_2$ is a set with the structure of a model of L_1 and the structure of a model of L_2, such that "the operations commute"
- Useful in computer science, for studying "algebraic effects"

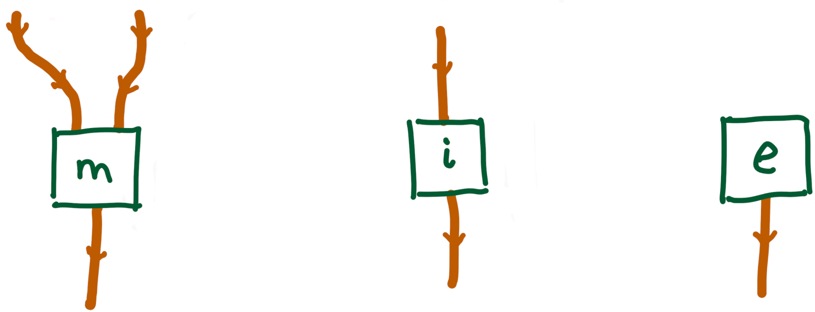
First half of project

- Tensor products of large Lawvere theories don't always exist
- Goncharov and Schröder* show they exist if one of the theories is "uniform"
- Corrected proof
- Applied to "Select" theory

* Powermonads and Tensors of Unranked Effects

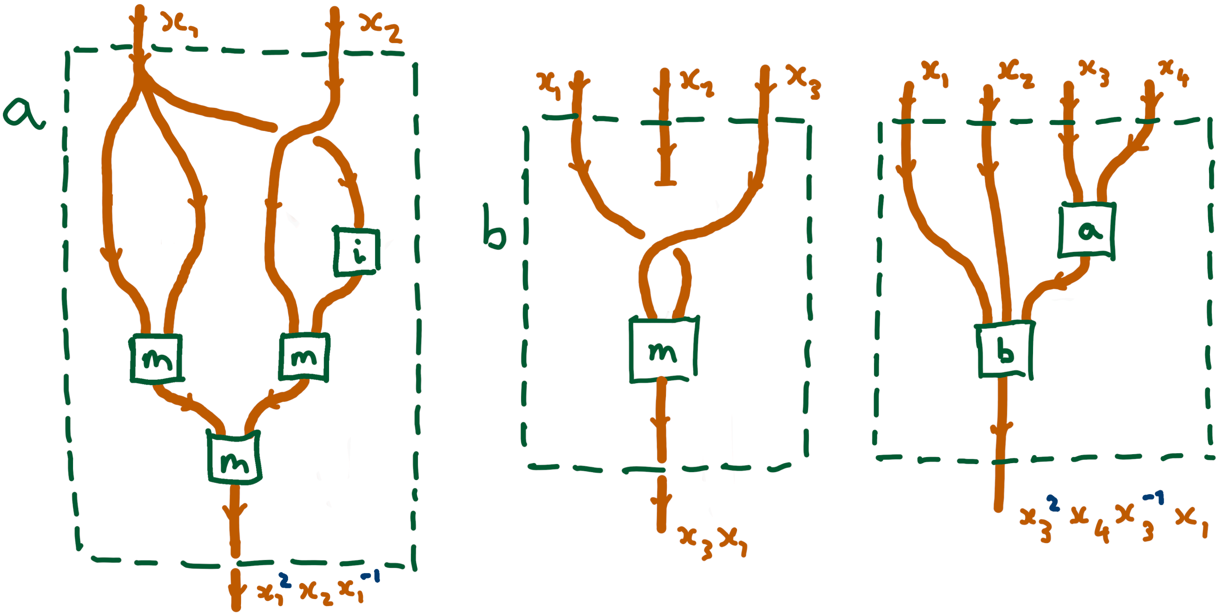
Notions of composition

- Generalised algebraic theory – have a collection of operations, and you can compose operations
- What do we want "you can compose operations" to mean?
- Think of an n-ary operation as a machine with n input wires and one output wire



Notions of composition

- To compose operations, hook up the wires
- In a Lawvere theory, wires can cross, split and end



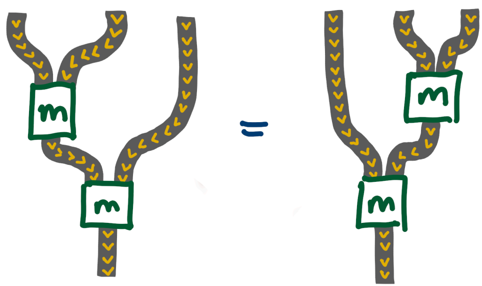
Notions of composition

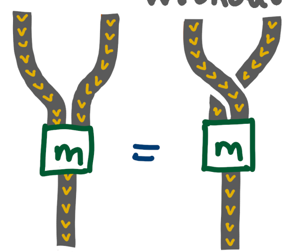
- When composing the operations of a (symmetric) operad, wires can't end or split
- We can picture this as machines joined by conveyor belts
- An operad can be specified by an equational theory where all variables in an equation occur exactly once on each side

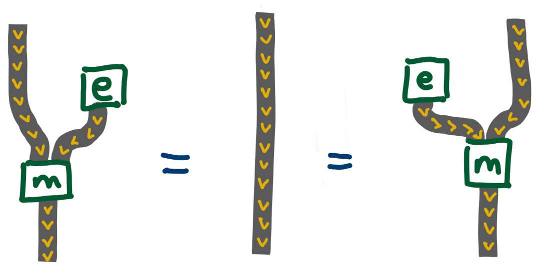
Notions of composition

- For example, there is a commutative monoid operad

(abelian group without inverses)







Notions of composition

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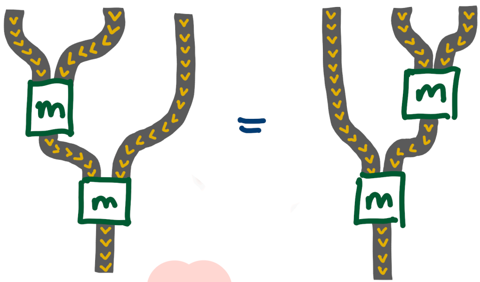

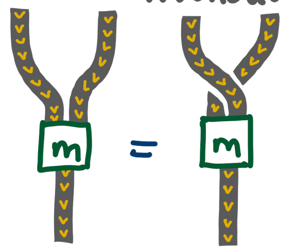

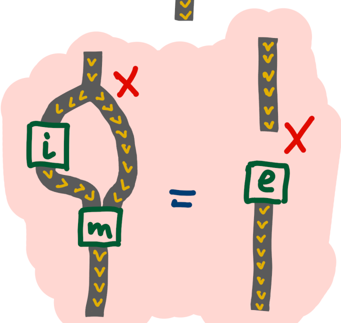

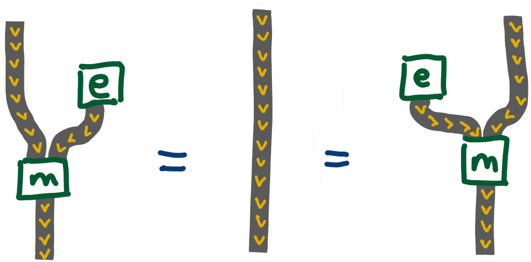

Second half of project

- Hyland* gives a definition of a generalised algebraic theory (using profunctors)

- Kock** gives a very different approach to operads (using polynomial functors over groupoids)

- Can we rigorously relate these approaches?

- Made some progress, more to be done

* Elements of a theory of algebraic theories

** Data Types with Symmetries and Polynomial Functors over Groupoids

Questions?